Matched Circles

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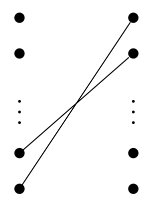
Strands Algebra $\mathcal{A}(n)$

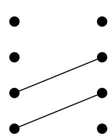
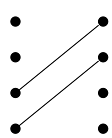
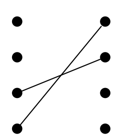
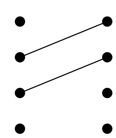
Definition.

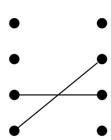
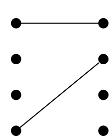
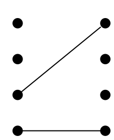
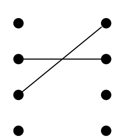
- $\mathcal{A}(n,k)$ is the algebra over $\mathbb{F}_2$ of diagrams

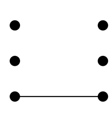
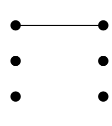
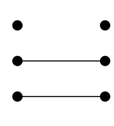
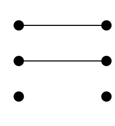
  

  of n vertices and k strands, where strands go up and are considered up to homotopy, multiplication is concatenation, and mismatches or double crossings become zero.
- The *strands algebra* is $\mathcal{A}(n) = \bigoplus_{k=0}^{n} \mathcal{A}(n,k)$
- $\partial(\text{diagram}) = \sum_{\text{crossing } c}(\text{diagram with } c \text{ smoothed})$

Example. For

$a =$  $b =$  $c =$  $d =$ 

$e =$  $+$  $f =$  $+$ 

$g =$  $+$  $+$  $+$ 

we have

$$ad = b, \qquad af = c, \qquad ag = a, \qquad a(\text{else}) = 0,$$

and

$$\partial(c) = b, \qquad \partial(e) = a, \qquad \partial(f) = d, \qquad \partial(\text{else}) = 0.$$

Proposition. $\mathcal{A}(n,k)$ is a differential algebra.

Proof. Consider the algebra $\overline{\mathcal{A}}(n,k)$ of diagrams where double crossings are allowed, which is a differential algebra under the same ∂:

- the product is associative
- $\partial^2(\text{diagram}) = 2\sum_{\text{crossings } c_1, c_2}(\text{diagram with } c_1, c_2 \text{ smoothed}) = 0$
- ∂ satisfies the Leibniz rule.

Consider the subspace $\overline{\mathcal{A}}_d(n,k)$ spanned by diagrams with at least one double crossing, where

$$\mathcal{A}(n,k) = \overline{\mathcal{A}}(n,k)/\overline{\mathcal{A}}_d(n,k)$$

as subspaces. It suffices to show $\overline{\mathcal{A}}_d(n,k)$ is a differential ideal.

Consider a diagram in $\overline{\mathcal{A}}_d(n,k)$. If it has at least two double crossings, then certainly $\partial(\text{diagram}) \in \overline{\mathcal{A}}_d(n,k)$. If it has only one double crossing c_1, c_2, then

$$\partial(\text{diagram}) = \sum_{\text{crossing } c \neq c_1, c_2} (c \text{ smoothed}) + (c_1 \text{ smoothed}) + (c_2 \text{ smoothed}),$$

where the terms in the sum are in $\overline{\mathcal{A}}_d(n,k)$ and where the last two terms are equal since strands are considered up to homotopy. □

Matched Circles

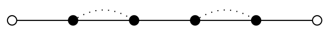
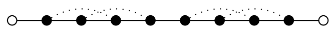
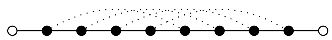
Definition. A *pointed matched circle* $\mathcal{Z} = (Z, \mathfrak{a}, M, z)$ is the following data:

- an oriented circle Z
- $4k$ points $\mathfrak{a}$
- a pairing M of the points in $\mathfrak{a}$ such that performing on the $2k$ pairs of points yields a single circle
- a basepoint z.

We will largely be ignoring the basepoint and the orientation of Z.

Remark. Attaching 2-dimensional 1-handles according to M gives an oriented surface $F(\mathcal{Z})$. We think of $\mathcal{Z}$ as parametrizing the surface $F(\mathcal{Z})$.

Example. Here are some examples of pointed matched circles, where the basepoint and orientation are omitted and where the endpoints are identified:

-  corresponds to a torus
-  is illegal since surgery yields two circles
-  is the genus 2 surface
-  is also the genus 2 surface

Thinking of the points $\mathfrak{a}$ as the vertices in diagrams in $\mathcal{A}(4k)$, we now demand that our diagrams be compatible with M: for each matched pair of vertices, at most one of the vertices can be touched by a strand. If a compatible diagram has h horizontal strands, the *smear* of the diagram is the sum over the 2^h possible diagrams with the same upward strands but with the obvious freedom in the horizontal strands. For example e, f, g are smears of any of their summands.

Definition. The *algebra* $\mathcal{A}(\mathcal{Z})$ *associated to a pointed matched circle* $\mathcal{Z}$ is the subalgebra of $\mathcal{A}(4k)$ generated by smears of compatible diagrams. Write

$$\mathcal{A}(\mathcal{Z}, i) = \mathcal{A}(\mathcal{Z}) \cap A(4k, k+i)$$

(this will be convenient later) so that

$$\mathcal{A}(\mathcal{Z}) = \bigoplus_{i=-k}^{k} \mathcal{A}(\mathcal{Z}, i).$$

Example. The elements of $\mathcal{A}(\text{torus}, 0)$ are the 6 single upward strands along with the two idempotents. The elements a, b, c, d, e, f, g are the elements of $\mathcal{A}(\text{torus}, 1)$; one can see this by doing casework on the number of horizontal strands. Note that

$$H_*(\mathcal{A}(\text{torus}, 1)) = \ker \partial / \operatorname{im} \partial = \mathbb{F}_2$$

is generated by g. In fact $H_*(\mathcal{A}(\text{genus } g \text{ surface}, g)) = \mathbb{F}_2$.

Looking ahead

Let Y be a 3-manifold where $F = \partial Y$ has a pointed matched circle $\mathcal{Z}$. Using this data, we will construct the *bordered Heegaard Floer invariants of* Y, which consist roughly speaking of the following two things:

- *the type D module* $\widehat{CFD}(Y)$ *of* Y, a left differential module over $\mathcal{A}(-\mathcal{Z})$
- *the type A module* $\widehat{CFA}(Y)$ *of* Y, a right A_∞-module over $\mathcal{A}(\mathcal{Z})$

We wish to compute the Heegaard Floer homology $\widehat{HF}(Y)$ of Y. Sarkar-Wang showed that this can be computed algorithmically. The following so-called Pairing Theorem is hoped to make this even more efficient:

Theorem. *Let* Y_1, Y_2 *be 3-manifolds where* $\partial Y_1 = F$ *and* $\partial Y_2 = -F$*, and let* $Y = Y_1 \cup_F Y_2$*. Then*

$$\widehat{HF}(Y) = H_*\left(\widehat{CFA}(Y_1) \, \widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \, \widehat{CFD}(Y_2)\right).$$

This lets us divide-and-conquer Y.